

STABILITY OF GEODESIC SPHERES IN \mathbb{S}^{n+1} UNDER CONSTRAINED CURVATURE FLOWS

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ABSTRACT. In this paper we discuss the stability of geodesic spheres in \mathbb{S}^{n+1} under constrained curvature flows. We prove that under some standard assumptions on the speed and weight functions, the spheres are stable under perturbations that preserve a volume type quantity. This extends results of [3] and [7] to a Riemannian manifold setting.

1. INTRODUCTION

We consider a family of hypersurfaces that are compact without boundary, $\{\Omega_t\}_{t \in [0, T]}$, inside a Riemannian manifold (N^{n+1}, \bar{g}) moving with a speed function \hat{G} in the direction of the normal. If Ω_0 is given by an embedding $\tilde{X}_0 : M^n \rightarrow N^{n+1}$ then the family is obtained by solving for an $X : M^n \times [0, T] \rightarrow N^{n+1}$ that satisfies

$$(1) \quad \frac{\partial X}{\partial t} = \hat{G}(X)\nu, \quad X(\cdot, 0) = \tilde{X}_0$$

where ν is the outer unit normal to Ω_t , with $\Omega_t = X(M^n, t)$. We will consider speed functions, \hat{G} , of the form:

$$(2) \quad \hat{G}(X) := \frac{\int_{M^n} F(\kappa) \hat{\Xi}(X) d\mu}{\int_{M^n} \hat{\Xi}(X) d\mu} - F(\kappa),$$

where $d\mu$ and $\kappa = (\kappa_1, \dots, \kappa_n)$ are, respectively, the induced volume form and principal curvatures (i.e. eigenvalues of the Weingarten map \mathcal{W}) on $X(\Omega)$, $\hat{\Xi}$ is a weight function, and F is a smooth, symmetric function on \mathbb{R}^n satisfying $\frac{\partial F}{\partial \kappa_a}(\kappa(\tilde{X}_0)) > 0$.

We will consider a fairly general form of the weight function $\hat{\Xi}$, however of special interest are the cases when

$$(3) \quad \hat{\Xi}(X) = \sum_{a=0}^{n+1} c_a \hat{\Xi}_a(X)$$

for some $\{c_0, \dots, c_{n+1}\} \in \mathbb{R}^{n+2}$, where

$$(4) \quad \hat{\Xi}_a(X) := \begin{cases} -\frac{g^{ik}}{(n+1)\binom{n}{n-a}} \left(\frac{\partial E_{n-a}}{\partial h_j^i} \bar{g}(\bar{R}(\nu, T_k)\nu, T_j) + \nabla_j \nabla_k \left(\frac{\partial E_{n-a}}{\partial h_j^i} \right) \right) + \binom{n+1}{n+1-a}^{-1} E_{n+1-a} & \text{if } a = 0, \dots, n, \\ 1 & \text{if } a = n+1, \end{cases}$$

g is the induced metric on $X(M^n)$, $\bar{R}(U, W)Z = \bar{\nabla}_W \bar{\nabla}_U Z - \bar{\nabla}_U \bar{\nabla}_W Z + \bar{\nabla}_{[U, W]} Z$ is the Riemann curvature tensor of (N^{n+1}, \bar{g}) , $\bar{\nabla}$ and ∇ are the Levi-Civita connections on (N^{n+1}, \bar{g}) and induced on $(X(M^n), g)$ respectively, and

$$E_a := \sum_{1 \leq b_1 < \dots < b_a \leq n} \prod_{i=1}^a \kappa_{b_i},$$

are the elementary symmetric functions of the Weingarten map. Note that for hypersurfaces in Euclidean space $g^{ik}\nabla_k\left(\frac{\partial E_{n-a}}{\partial h_j^i}\right) = 0$; see the proof of Lemma 2.2.2. in [6]. With $\hat{\Xi}$ as in (3), the flow (1) preserves the real valued quantity

$$(5) \quad \hat{V}(\Omega) := \sum_{a=0}^{n+1} c_a \hat{V}_a(\Omega),$$

where \hat{V}_a are the mixed volumes

$$\hat{V}_a(\Omega) := \begin{cases} \frac{1}{(n+1)\binom{n}{n-a}} \int_{M^n} E_{n-a} d\mu & \text{if } a = 0, \dots, n, \\ \text{Vol}(\Omega) & \text{if } a = n+1. \end{cases}$$

The topic of intrinsic volumes is more complicated in spherical space than Euclidean space. For example, in [4] they consider three different definitions, however each is a linear combination of the mixed volumes defined above and hence can be preserved by choosing the c_a constants appropriately. See Appendix A for a proof that \hat{V} is preserved under the flow when $\hat{\Xi}$ is given by (3).

This flow in Euclidean space, and with a weight function such that a mixed volume is preserved, has been studied previously by McCoy in [11]. There it was proved that under some additional conditions on F , for example homogeneity of degree one and convexity or concavity, initially convex hypersurfaces admit a solution for all time and that the hypersurfaces converge to a sphere as $t \rightarrow \infty$. This was an extension of a result by Huisken [9] who proved the result for the volume preserving mean curvature flow (VPMCF). The stability of spheres has previously been considered by Escher and Simonett in [3] for the case of the VPMCF in Euclidean space where it was proved that they were stable under small perturbations in the little Hölder space $h^{1,\alpha}$, any $\alpha \in (0, 1)$. This result was extended by the author, [8], to the case of mixed-volume preserving curvature flows, with the perturbations in the space $h^{2,\alpha}$ to account for the fully nonlinear nature of the flows.

For flows of this nature in Riemannian manifolds Huisken noted in [9] that even the VPMCF in \mathbb{S}^{n+1} will, in general, not preserve the convexity of a hypersurface, thus making the standard analysis more difficult. In [1] Alikakos and Freire prove that if the manifold N^{n+1} has a finite number of critical points of the scalar curvature that are all non-degenerate, then the geodesic spheres close to the critical points are stable under volume preserving perturbations. In the case when N^{n+1} is hyperbolic space, Cabezas-Rivas and Vicente in [2] prove that the VPMCF of hypersurfaces that satisfy a certain convexity property exist for all time and converge to geodesic spheres. They also prove that geodesic spheres in these manifolds are stable with respect to the VPMCF under $h^{1,\alpha}$ perturbations.

In this paper we consider the stability of geodesic spheres in \mathbb{S}^{n+1} under the flow (1). The main theorem is:

Theorem 1.1. *A geodesic sphere $\mathcal{S} \subset \mathbb{S}^{n+1}$ is stable under perturbations in $h^{2,\alpha}$, for any $\alpha \in (0, 1)$, with respect to the flow (1), with \hat{G} as in (2), if the following hold:*

- F is a smooth, symmetric function of the principal curvatures,
- $\frac{\partial F}{\partial \kappa_1}(\kappa(\mathcal{S})) > 0$, and
- $\hat{\Xi}(\mathcal{S}) = \text{const} \neq 0$.

To be precise let Ω_0 be a $h^{2,\alpha}$ -close normal geodesic graph over \mathcal{S} , then the flow by (1) exists for all time and the hypersurfaces $\Omega_t := X(M^n, t)$ converge in $h^{2,\alpha}$ to a geodesic sphere close to \mathcal{S} .

Note that the little Hölder spaces on a manifold, $h^{k,\alpha}(M^n)$ for $k \in \mathbb{N}_0$ and $\alpha \in (0, 1)$, are defined as the completion of the smooth functions inside the standard Hölder space $C^{k,\alpha}(M^n)$. They are useful in analysing stability properties as they obey a self interpolation property, [5, Equation 19].

The paper is structured as follows. In Section 2 we consider the properties of hypersurfaces that are normal geodesic graphs over a base hypersurface. We also give an equation on the space of graph functions that is equivalent to (1). Section 3 derives the linearisation of the speed function in the case where the hypersurfaces are graphs over a geodesic sphere in \mathbb{S}^{n+1} . The space of functions that define geodesic spheres close to the base sphere is then analysed in Section 4, and finally Theorem 1.1 is proved in Section 5. In Appendix A we derive the formula for the weight function so that the quantity \hat{V} is preserved under the flow.

2. NORMAL GEODESIC GRAPHS AND EQUIVALENT EQUATIONS

We will consider the situation where the hypersurfaces are normal (geodesic) graphs over a base hypersurface $X_0(M^n)$ with outer unit normal ν_0 . These hypersurfaces can be written as $X_u(p) = \gamma(p, u(p))$ where $u : M^n \rightarrow \mathbb{R}$, and $\gamma_p(s) := \gamma(p, s)$ is the unique unit speed geodesic satisfying $\gamma_p(0) = X_0(p)$ and $\dot{\gamma}_p(0) = \nu_0(p)$, we use a dot to denote a derivative with respect to the geodesic parameter, s . Note that we require $\|u\|_{C^0} < C_0$, where C_0 is the injectivity radius of $X_0(M^n) \subset N^{n+1}$.

Standard calculations give us the following formulas

Lemma 2.1. *The tangent vectors for a normal graph $X_u(M^n)$ are given by*

$$T_i(u) = \frac{\partial \gamma}{\partial p^i} + \nabla_i u \dot{\gamma} \Big|_{s=u},$$

the induced metric components are given by

$$g_{ij}(u) = \bar{g} \left(\frac{\partial \gamma}{\partial p^i}, \frac{\partial \gamma}{\partial p^j} \right) \Big|_{s=u} + \nabla_i u \nabla_j u,$$

and the unit normal by

$$\nu(u) = \sqrt{1 - |\nabla u|_{g(u)}^2} \dot{\gamma} - \frac{g^{ij}(u) \nabla_i u}{\sqrt{1 - |\nabla u|_{g(u)}^2}} \frac{\partial \gamma}{\partial p^j} \Big|_{s=u},$$

where $g^{ij}(u)$ are the components of the inverse of $g(u)$, also note that $|\nabla u|_{g(u)} = \sqrt{g^{ij}(u) \nabla_i u \nabla_j u} < 1$.

Proof. The formula for the tangent vectors follows directly from $X_u(p) = \gamma(p, u(p))$. The metric formula then follows from using the unit speed condition $\bar{g}(\dot{\gamma}, \dot{\gamma}) = 1$ and the formula $\bar{g}(\frac{\partial \gamma}{\partial p^i}, \dot{\gamma}) = 0$. In fact,

$$\frac{\partial}{\partial s} \left(\bar{g} \left(\frac{\partial \gamma}{\partial p^i}, \dot{\gamma} \right) \right) = \bar{g} \left(\bar{\nabla}_{\frac{\partial \gamma}{\partial p^i}} \dot{\gamma}, \dot{\gamma} \right) + \bar{g} \left(\frac{\partial \gamma}{\partial p^i}, \bar{\nabla}_{\dot{\gamma}} \dot{\gamma} \right) = 0,$$

where $\bar{\nabla}_V U = V(U) + \bar{\Gamma}_{\beta\gamma}^\alpha V^\beta U^\gamma \partial_\alpha$ is the Levi-Civita connection on (N^{n+1}, \bar{g}) , and we have used the space derivative of the unit speed condition and the geodesic condition. Hence $\bar{g}(\frac{\partial \gamma}{\partial p^i}, \dot{\gamma}) = \bar{g}(\frac{\partial \gamma}{\partial p^i}, \dot{\gamma}) \Big|_{s=0} = \bar{g}(\frac{\partial X_0}{\partial p^i}, \nu_0) = 0$, so

$$g_{ij}(u) = \bar{g}(T_i(u), T_j(u)) = \bar{g} \left(\frac{\partial \gamma}{\partial p^i}, \frac{\partial \gamma}{\partial p^j} \right) + \nabla_i u \nabla_j u \bar{g}(\dot{\gamma}, \dot{\gamma}) \Big|_{s=u}.$$

The formula for the unit normal can be seen by taking its inner product with the tangent vectors and using that $\bar{g}\left(\frac{\partial \gamma}{\partial p^i}, \frac{\partial \gamma}{\partial p^j}\right)\Big|_{s=u} = g_{ij}(u) - \nabla_i u \nabla_j u$. \square

We now aim to show that the equation (1) is equivalent to an equation on $C^2(M^n)$. We define $L(u) := \bar{g}(\dot{\gamma}|_{s=u}, \nu(u))^{-1} = \frac{1}{\sqrt{1-|\nabla u|_{g(u)}^2}}$ and $G(u) := L(u)\hat{G}(X_u)$, and consider the flow

$$(6) \quad \frac{\partial u}{\partial t} = G(u), \quad u(0) = u_0,$$

then we have

$$\frac{\partial X_u}{\partial t} = \frac{\partial u}{\partial t} \dot{\gamma}|_{s=u} = G(u) \dot{\gamma}|_{s=u},$$

and in particular

$$\left\langle \frac{\partial X_u}{\partial t}, \nu(u) \right\rangle = \hat{G}(X_u).$$

Therefore there is a tangential diffeomorphism $\phi_t : M^n \rightarrow M^n$, with $\phi_0 = id$, such that $X_u(\phi_t(p), t)$ satisfies (1), with initial embedding X_{u_0} . Likewise if X satisfies (1) and $X = X_u$ for some $u : M^n \times [0, T] \rightarrow \mathbb{R}$ then from $\frac{\partial X_u}{\partial t} = \frac{\partial u}{\partial t} \dot{\gamma}|_{s=u}$ we obtain, via the inner product with $\nu(u)$, that $\hat{G}(X_u) = L(u)^{-1} \frac{\partial u}{\partial t}$ and hence u satisfies (6). Therefore equations (6) and (1) are equivalent.

Note that because (6) is equivalent to (1), up to a tangential diffeomorphism, when $\hat{\Xi}$ is given by (3) we have that $V(u) := \hat{V}(X_u(M^n))$ is also a preserved quantity for (6). This can also be seen by using equation (17) in Appendix A to calculate the linearisation of $V(u)$:

$$\begin{aligned} DV(u)[w] &= D\hat{V}(X_u)[DX_u[w]] \\ &= D\hat{V}(X_u)[\dot{\gamma}|_{s=u}w] \\ &= \int_{M^n} \hat{\Xi}(X_u) \bar{g}(\dot{\gamma}|_{s=u}w, \nu(u)) d\mu_u \\ (7) \quad &= \int_{M^n} \hat{\Xi}(X_u) L(u)^{-1} w d\mu_u. \end{aligned}$$

Now by setting $w = \frac{\partial u}{\partial t} = L(u)\hat{G}(X_u)$ and using the form of \hat{G} in (2) we obtain $\frac{\partial V}{\partial t} = DV(u)\left[\frac{\partial u}{\partial t}\right] = 0$.

3. LINEARISATION ABOUT GEODESIC SPHERES IN \mathbb{S}^{n+1}

We start by giving some standard linearisation formulas, and we suppress that quantities are to be evaluated at X ,

Lemma 3.1. *The components of the metric of $X(M^n)$, $g_{ij}(X) := \bar{g}\left(\frac{\partial X}{\partial p^i}, \frac{\partial X}{\partial p^j}\right)$, have the linearisation*

$$Dg_{ij}(X)[Y] = \bar{g}\left(\bar{\nabla}_{T_i} Y, T_j\right) + \bar{g}\left(\bar{\nabla}_{T_j} Y, T_i\right),$$

where $T_i(X) := \frac{\partial X}{\partial p^i}$, and $\bar{\nabla}_{T_i} Y = \frac{\partial Y}{\partial p^i} + \bar{\Gamma}_{\beta\gamma}^\alpha Y^\beta T_i^\gamma \partial_\alpha$. The volume element $\mu(X) := \sqrt{\det(g(X))}$ has the linearisation

$$D\mu(X)[Y] = g^{ij} \bar{g}\left(\bar{\nabla}_{T_i} Y, T_j\right) \mu,$$

the unit normal of $X(M^n)$ satisfies

$$D\nu(X)[Y] + \bar{\Gamma}_{\beta\gamma}^\alpha \nu^\beta Y^\gamma \partial_\alpha = -g^{ij} \bar{g}\left(\bar{\nabla}_{T_i} Y, \nu\right) T_j,$$

the linearisation of the components of the second fundamental form, $h_{ij}(X) := \bar{g}(\bar{\nabla}_{T_i(X)} \nu(X), T_j(X))$, is

$$Dh_{ij}(X)[Y] = -\bar{g}(\bar{\nabla}_{T_i} \bar{\nabla}_{T_j} Y, \nu) - \bar{g}(\bar{R}(Y, T_i) \nu, T_j),$$

and finally the linearisation of the elements of the Weingarten map, $h_j^i(X) = g^{ik}(X)h_{kj}(X)$, are given by

$$Dh_j^i(X)[Y] = -g^{ik} \left(h_j^l \left(\bar{g}(\bar{\nabla}_{T_k} Y, T_l) + \bar{g}(\bar{\nabla}_{T_l} Y, T_k) \right) + \bar{g}(\bar{\nabla}_{T_k} \bar{\nabla}_{T_j} Y, \nu) + \bar{g}(\bar{R}(Y, T_k) \nu, T_j) \right).$$

Proof. The variations of the metric and volume element are straight from the definitions. By taking the linearisations of the formulas

$$\bar{g}(\nu, T_i) = 0, \text{ and } \bar{g}(\nu, \nu) = 1,$$

to obtain

$$\bar{g} \left(D\nu(X)[Y] + \bar{\Gamma}_{\beta\gamma}^\alpha \nu^\beta Y^\gamma \frac{\partial}{\partial q^\alpha}, T_i \right) + \bar{g}(\nu, \bar{\nabla}_{T_i} Y) = 0, \text{ and } 2\bar{g} \left(D\nu(X)[Y] + \bar{\Gamma}_{\beta\gamma}^\alpha \nu^\beta Y^\gamma \frac{\partial}{\partial q^\alpha}, \nu \right) = 0,$$

we are able to conclude the formula for the linearisation of the unit normal.

Now we consider the linearisation of the second fundamental form:

$$\begin{aligned} Dh_{ij}(X)[Y] &= \bar{g} \left(D(\bar{\nabla}_{T_i} \nu)(X)[Y] + \bar{\Gamma}_{\beta\gamma}^\alpha (\bar{\nabla}_{T_i} \nu)^\beta Y^\gamma \frac{\partial}{\partial q^\alpha}, T_j \right) + \bar{g}(\bar{\nabla}_{T_i} \nu, \bar{\nabla}_{T_j} Y) \\ &= \bar{g} \left(\bar{\nabla}_{T_i} \left(D\nu(X)[Y] + \bar{\Gamma}_{\beta\gamma}^\alpha \nu^\beta Y^\gamma \frac{\partial}{\partial q^\alpha} \right) + \bar{R}(T_i, Y) \nu, T_j \right) + \bar{g}(\bar{\nabla}_{T_i} \nu, \bar{\nabla}_{T_j} Y) \\ &= \bar{g}(\bar{\nabla}_{T_i} (-g^{kl} \bar{g}(\bar{\nabla}_{T_k} Y, \nu) T_l), T_j) - \bar{g}(\bar{R}(Y, T_i) \nu, T_j) + \bar{g}(\bar{\nabla}_{T_i} \nu, \bar{\nabla}_{T_j} Y) \\ &= -\bar{g}(\bar{\nabla}_{T_i} \bar{\nabla}_{T_j} Y, \nu) - \bar{g}(\bar{R}(Y, T_i) \nu, T_j). \end{aligned}$$

Finally we use that $h_j^i = g^{ik}h_{kj}$ and that $Dg^{ik}(X)[Y] = -g^{ip}g^{kq}Dg_{pq}(X)[Y]$ to obtain the linearisation of the Weingarten map components

$$\begin{aligned} Dh_j^i(X)[Y] &= -g^{ip}g^{kq} \left(\bar{g}(\bar{\nabla}_{T_p} Y, T_q) + \bar{g}(\bar{\nabla}_{T_q} Y, T_p) \right) h_{kj} - g^{ik} \left(\bar{g}(\bar{\nabla}_{T_k} \bar{\nabla}_{T_j} Y, \nu) + \bar{g}(\bar{R}(Y, T_k) \nu, T_j) \right) \\ &= -g^{ik} \left(h_j^l \left(\bar{g}(\bar{\nabla}_{T_k} Y, T_l) + \bar{g}(\bar{\nabla}_{T_l} Y, T_k) \right) + \bar{g}(\bar{\nabla}_{T_k} \bar{\nabla}_{T_j} Y, \nu) + \bar{g}(\bar{R}(Y, T_k) \nu, T_j) \right). \end{aligned}$$

□

Lemma 3.2.

$$D\hat{G}(X)[Y] = \frac{\int_{M^n} \sum_{a=1}^n \frac{\partial F}{\partial \kappa_a}(\kappa(X)) D\kappa_a(X)[Y] \hat{\Xi}(X) - \frac{\hat{G}(X)}{\hat{\mu}(X)} D(\hat{\Xi}\hat{\mu})(X)[Y] d\mu}{\int_{M^n} \hat{\Xi}(X) d\mu} - \sum_{a=1}^n \frac{\partial F}{\partial \kappa_a}(\kappa(X)) D\kappa_a(X)[Y],$$

$$D\kappa_a(X_0)[w\nu_0] = -\hat{\zeta}_a^i \hat{\zeta}_a^j \left(\hat{\nabla}_i \hat{\nabla}_j w + \bar{g}(\bar{R}(\nu_0, \hat{T}_i) \nu_0, \hat{T}_j) w \right) - \hat{\kappa}_a^2 w,$$

where \hat{T}_i are tangent vectors, $\hat{\nabla}$ is the Levi-Civita connection, and $\hat{\zeta}_a$ is the principle direction (eigenvector of the Weingarten map \hat{W}) corresponding to the principle curvature $\hat{\kappa}_a$ of $X_0(M^n)$. In particular if $X_0(M^n)$ is totally umbilic we have

$$\sum_{a=1}^n \frac{\partial F}{\partial \kappa_a}(\hat{\kappa}) D\kappa_a(X_0)[w\nu_0] = -\frac{\partial F}{\partial \kappa_1}(\hat{\kappa}) \left(\Delta_{\hat{g}} w + |\hat{W}|_{\hat{g}}^2 w + \bar{R}ic(\nu_0, \nu_0) w \right),$$

where \hat{g} is the metric of $X_0(M^n)$.

Proof. The first formula follows directly from the definition of $\hat{G}(X)$, while the second formula follows from $D\kappa_a(X)[Y] = \zeta^i(X)\zeta^j(X)g_{jk}Dh_i^k(X)[Y]$ and Lemma 3.1. □

Now we consider $N^{n+1} = \mathbb{S}^{n+1}$ with coordinates q_α , $\alpha = 1, \dots, n+1$, with $q_1 \in [0, 2\pi)$ and $q_\alpha \in [0, \pi)$ for $\alpha = 2, \dots, n+1$, such that the metric is

$$\bar{g} = g_{\mathbb{S}^{n+1}} = \sum_{\alpha=1}^{n+1} \prod_{\beta=\alpha+1}^{n+1} \sin(q_\beta)^2 dq^\alpha{}^2,$$

and $X_0(M^n)$ equal to the n -sphere $q_{n+1} = \theta$, for some fixed $\theta \in (0, \pi)$, which we denote S_θ . With this set up normal graphs take the form

$$(8) \quad X_u(p) = (p_1, \dots, p_n, \theta + u(p)),$$

where $p = (p_1, \dots, p_n) \in \mathbb{S}^n$ and $C_0 = \min(\theta, \pi - \theta)$. The tangent vectors, metric, and Weingarten map for $X_0(M^n)$ are then

$$\begin{aligned} \dot{T}_i &= \delta_i^\alpha \frac{\partial}{\partial q^\alpha}, \\ \dot{g}_{ij} &= \sin(\theta)^2 g_{\mathbb{S}^n} = \sin(\theta)^2 \sum_{i=1}^n \prod_{j=i+1}^n \sin(p_j)^2 dp^{i^2}, \\ \mathring{W} &= \cot(\theta) Id. \end{aligned}$$

Lemma 3.3.

$$\Xi_a(X_0) = \begin{cases} \frac{\cot(\theta)^{n-a-1}}{n+1} (a \cot(\theta)^2 - n + a) & \text{if } a = 0, \dots, n, \\ 1 & \text{if } a = n+1. \end{cases}$$

Proof. This follows from a straightforward calculation using (3). Firstly we note that since $\mathring{\nabla}_k \mathring{h}_j^i = 0$ we have

$$\mathring{\nabla}_j \mathring{\nabla}_k \left(\frac{\partial E_{n-a}}{\partial h_j^i}(\mathring{k}) \right) = \mathring{\nabla}_j \left(\frac{\partial^2 E_{n-a}}{\partial h_j^i \partial h_q^p}(\mathring{k}) \mathring{\nabla}_k \mathring{h}_q^p \right) = 0.$$

Next we use that $\bar{g}(\bar{R}(v_0, \dot{T}_k)v_0, \dot{T}_j) = \dot{g}_{kj}$, $E_a(\mathring{k}) = \binom{n}{a} \cot(\theta)^a$, and the formula (see Proposition B.0.2. in [6] for example)

$$(9) \quad \frac{\partial E_a}{\partial h_j^i}(\kappa) = \sum_{b=0}^{a-1} (-1)^b (\mathcal{W}^b)_i^j E_{a-1-b}(\kappa),$$

to calculate the remaining terms:

$$\begin{aligned} \Xi_a(X_0) &= \frac{-\dot{g}^{ik}}{(n+1)\binom{n}{n-a}} \frac{\partial E_{n-a}}{\partial h_j^i}(\mathring{k}) \dot{g}_{kj} + \frac{E_{n-a+1}(\mathring{k})}{\binom{n+1}{n-a+1}} \\ &= \frac{-\delta_j^i}{(n+1)\binom{n}{n-a}} \sum_{b=0}^{n-a-1} (-1)^b (\mathring{W}^b)_i^j E_{n-a-1-b}(\mathring{k}) + \frac{\binom{n}{n+1-a} \cot(\theta)^{n+1-a}}{\binom{n+1}{n+1-a}} \\ &= \frac{-1}{(n+1)\binom{n}{n-a}} \sum_{b=0}^{n-a-1} (-1)^b \text{tr}(\mathring{W}^b) \binom{n}{n-a-1-b} \cot(\theta)^{n-a-1-b} + \frac{a}{n+1} \cot(\theta)^{n+1-a} \\ &= \frac{-n}{(n+1)\binom{n}{n-a}} \sum_{b=0}^{n-a-1} (-1)^b \binom{n}{n-a-1-b} \cot(\theta)^{n-a-1} + \frac{a}{n+1} \cot(\theta)^{n+1-a} \\ &= \frac{a-n}{n+1} \cot(\theta)^{n-a-1} + \frac{a}{n+1} \cot(\theta)^{n+1-a}. \end{aligned}$$

□

- Remark 3.4.** • For our purposes the important thing here is that each $\Xi_a(X_0)$ is a constant and, hence, the form (3) will be allowable in our theorem, provided $c_a \in \mathbb{R}$, $a = 0, \dots, n+1$, are such that $\hat{\Xi}(X_0) \neq 0$.
- It should be noted that we have $\Xi_a(X_0) = 0$ if and only if $\theta \in \{\theta_a, \frac{\pi}{2}, \pi - \theta_a\}$, where $\theta_a = \arcsin\left(\sqrt{\frac{a}{n}}\right)$, except in the cases of $a = n-1$ when it is if and only if $\theta \in \{\theta_a, \pi - \theta_a\}$ and $a = n+1$ when it is never zero.
 - As suggested in [4] a better intrinsic volume to use in spherical space may be

$$\hat{U}_a(\Omega) = \frac{\Gamma(n+2)}{2^{n+2}\pi^{\frac{n}{2}}} \sum_{b=0}^{\lfloor \frac{n+1-a}{2} \rfloor} \frac{\hat{V}_{a+2b}(\Omega)}{\Gamma\left(\frac{a+2b}{2} + 1\right)\Gamma\left(\frac{n-a-2b}{2} + 1\right)},$$

where Γ is the usual Gamma function, for which the linearisation is

$$D\hat{U}_a(X)[Y] = \int_{M^n} \hat{Z}_a(X) \bar{g}(Y, \nu) d\mu,$$

where

$$\hat{Z}_a(X) = \frac{\Gamma(n+2)}{2^{n+2}\pi^{\frac{n}{2}}} \sum_{b=0}^{\lfloor \frac{n+1-a}{2} \rfloor} \frac{\hat{\Xi}_{a+2b}(X)}{\Gamma\left(\frac{a+2b}{2} + 1\right)\Gamma\left(\frac{n-a-2b}{2} + 1\right)}.$$

In this case the linearisation functions at X_0 have the simpler form

$$\hat{Z}_a(X_0) = \frac{a\Gamma(n+1)\cos(\theta)^{n+1-a}}{2^{n+2}\pi^{\frac{n}{2}}\Gamma\left(\frac{a}{2} + 1\right)\Gamma\left(\frac{n-a}{2} + 1\right)},$$

and is zero if and only if $\theta = \frac{\pi}{2}$ and $a \neq n+1$.

Lemma 3.5. We assume that $\hat{\Xi}(X_0) = \text{const} \neq 0$, then

$$DG(0)[w] = \frac{\partial F}{\partial \kappa_1}(\hat{\kappa}) \sin(\theta)^{-2} \left(\Delta_{g_{\mathbb{S}^n}} w + nw - \int_{\mathbb{S}^n} w d\mu_0 \right)$$

Proof. To calculate $DG(0)$ we first note that $DL(0)[w] = 0$ and $DX_u|_{u=0}[w] = w\nu_0$, so that

$$\begin{aligned} DG(0)[w] &= D\hat{G}(X_0)[w\nu_0] \\ &= \frac{\int_{\mathbb{S}^n} \sum_{a=1}^n \frac{\partial F}{\partial \kappa_a}(\hat{\kappa}) D\kappa_a(X_0)[w\nu_0] \hat{\Xi}(X_0) - \frac{\hat{G}(X_0)}{\hat{\mu}(X_0)} D(\hat{\Xi}\hat{\mu})(X_0)[w\nu_0] d\mu_0}{\int_{\mathbb{S}^n} \hat{\Xi}(X_0) d\mu_0} - \sum_{a=1}^n \frac{\partial F}{\partial \kappa_a}(\hat{\kappa}) D\kappa_a(X_0)[w\nu_0] \\ &= \frac{\partial F}{\partial \kappa_1}(\hat{\kappa}) \left(\Delta_{\hat{g}} w + |\hat{\mathcal{W}}|_{\hat{g}}^2 w + \bar{Ric}(\nu_0, \nu_0) w \right) - \int_{\mathbb{S}^n} \frac{\partial F}{\partial \kappa_1}(\hat{\kappa}) \left(\Delta_{\hat{g}} w + |\hat{\mathcal{W}}|_{\hat{g}}^2 w + \bar{Ric}(\nu_0, \nu_0) w \right) d\mu_0 \\ &= \frac{\partial F}{\partial \kappa_1}(\hat{\kappa}) \left(\Delta_{\hat{g}} w + |\hat{\mathcal{W}}|_{\hat{g}}^2 w + \bar{Ric}(\nu_0, \nu_0) w - \int_{\mathbb{S}^n} \Delta_{\hat{g}} w + |\hat{\mathcal{W}}|_{\hat{g}}^2 w + \bar{Ric}(\nu_0, \nu_0) w d\mu_0 \right) \\ &= \frac{\partial F}{\partial \kappa_1}(\hat{\kappa}) \left(\sin(\theta)^{-2} \Delta_{g_{\mathbb{S}^n}} w + n \cot(\theta)^2 w + nw - \int_{\mathbb{S}^n} n \cot(\theta)^2 w + nw d\mu_0 \right) \\ &= \frac{\partial F}{\partial \kappa_1}(\hat{\kappa}) \sin(\theta)^{-2} \left(\Delta_{g_{\mathbb{S}^n}} w + nw - n \int_{\mathbb{S}^n} w d\mu_0 \right) \end{aligned}$$

□

Corollary 3.6. Let $c_a \in \mathbb{R}$, $a = 0, \dots, n+1$, be such that $\hat{\Xi}(X_0) = \text{const} \neq 0$, and the smooth, symmetric function F be such that $\frac{\partial F}{\partial \kappa_1}(\hat{\kappa}) > 0$, then

$$\sup \{ \lambda : \lambda \in \sigma(DG(0)) \setminus \{0\} \} < 0,$$

and 0 is an eigenvalue of $DG(0)$ with multiplicity $n + 2$ and eigenfunctions given by the first order spherical harmonics on \mathbb{S}^n , $Y_1^{(n)}, \dots, Y_{n+1}^{(n)}$, and the constant function.

Remark 3.7. Our coordinates give the following formula for the spherical harmonics

$$Y_i^{(n)}(p) = \prod_{j=i}^n \sin(p_j) \cos(p_{j-1}).$$

4. SPACE OF GEODESIC SPHERES IN \mathbb{S}^{n+1}

In this section we consider how to parameterise the space of geodesic spheres as normal geodesic graphs over a particular sphere. That is we look to find the space of functions such that X_u is a sphere near X_0 for any u in the space, and that all spheres near X_0 are accounted for. To find the parametrisation we embed \mathbb{S}^{n+1} in \mathbb{R}^{n+2} using the first order spherical harmonics on \mathbb{S}^{n+1} :

$$Z(q) = (Y_1^{(n+1)}(q), \dots, Y_{n+2}^{(n+1)}(q)).$$

The image of X_0 lies in the plane $x_{n+2} = \cos(\theta)$, so any non-perpendicular sphere is determined by the values b_i , $i = 1, \dots, n + 2$, such that it lies in the plane

$$x_{n+2} + \sum_{i=1}^{n+1} b_i x_i = \sqrt{1 + |b|^2}(\cos(\theta) + b_{n+2}),$$

for $(b_1, \dots, b_{n+2}) \in \mathbb{R}^{n+1} \times (-1 - \cos(\theta), 1 - \cos(\theta))$, where $|b| = \left(\sum_{i=1}^{n+1} b_i^2\right)^{\frac{1}{2}}$. The radius of the sphere is given by $\tilde{R} = \tilde{R}(b_{n+2}) = \sqrt{\sin(\theta)^2 - 2 \cos(\theta) b_{n+2} - b_{n+2}^2}$. Using the form of the normal graph (8) we find that u satisfies

$$\cos(\theta + u(p)) + \sin(\theta + u(p)) \sum_{i=1}^{n+1} b_i \prod_{j=i}^n \sin(p_j) \cos(p_{j-1}) = \sqrt{1 + |b|^2}(\cos(\theta) + b_{n+2}),$$

so by using the formula for the spherical harmonics on \mathbb{S}^n and solving for u we obtain that the graph functions for the spheres are given by

$$(10) \quad u_b = \arctan \left(\frac{\sum_{i=1}^{n+1} b_i Y_i^{(n)} + \sqrt{1 + |b|^2}(\cos(\theta) + b_{n+2}) \sqrt{1 + \left(\sum_{i=1}^{n+1} b_i Y_i^{(n)}\right)^2 - (1 + |b|^2)(\cos(\theta) + b_{n+2})^2}}{(1 + |b|^2)(\cos(\theta) + b_{n+2})^2 - \left(\sum_{i=1}^{n+1} b_i Y_i^{(n)}\right)^2} \right) - \theta,$$

where \arctan is defined such that $\arctan : \mathbb{R} \rightarrow [0, \pi)$ and we require $|b|^2 < \frac{1}{(\cos(\theta) + b_{n+2})^2} - 1$. This requirement means that the geodesic spheres considered divide the poles $q_{n+1} = 0$ and $q_{n+1} = \pi$.

Note $u_0 = 0$ so this gives the base geodesic sphere, and if we linearise, with respect to the parameters, at the base sphere we have

$$Du_b|_{b=0}[z] = \sum_{j=1}^{n+1} z_j Y_j^{(n)} + \frac{1}{\sin(\theta)} z_{n+2}.$$

We have thus proved the following

Lemma 4.1. *The space of graph functions defining a sphere non-perpendicular to X_0 (M^n) and separating the poles is given by*

$$\mathcal{S} := \{u_b \in C^0(\mathbb{S}^n) : b \in \mathbb{R}^{n+1} \times (-1 - \cos(\theta), 1 - \cos(\theta)), |b|^2 < \frac{1}{(\cos(\theta) + b_{n+2})^2} - 1\}$$

where u_b is defined as in (10). Further, at $b = 0$ it has the tangent space $T_0\mathcal{S} = \text{span}(\{Y_i^{(n)} \in C^0(\mathbb{S}^n) : i = 1, \dots, n+1\} \cup \{1\})$ and \mathcal{S} is locally a differentiable graph over it.

5. PROOF OF MAIN THEOREM

The proof of Theorem 1.1 follows precisely as in [7] since $DG(0)$ is a positive multiple of the linear operator in that paper, see Lemma 3.1. of [7] for its definition, and the stationary solutions are again a graph over $\text{Null}(DG(0))$. Firstly, since $DG(0)[w] + \frac{\partial F}{\partial \kappa_1}(\hat{\kappa}) \sin(\theta)^{-2} \int_{\mathbb{S}^n} w d\mu_0$ is the negative of an elliptic operator, it is sectorial as a map from $h^{2,\alpha}(\mathbb{S}^n)$ to $h^{0,\alpha}(\mathbb{S}^n)$ for any $\alpha \in (0, 1)$. Next, we use that $\frac{\partial F}{\partial \kappa_1}(\hat{\kappa}) \sin(\theta)^{-2} \int_{\mathbb{S}^n} w d\mu_0$ is a bounded linear map from $h^{2,\alpha}(\mathbb{S}^n)$ to $h^{2,\alpha}(\mathbb{S}^n)$ to conclude that $DG(0) : h^{2,\alpha}(\mathbb{S}^n) \rightarrow h^{0,\alpha}(\mathbb{S}^n)$ is sectorial for any $\alpha \in (0, 1)$. Now, as being sectorial is a stable condition, this implies that $DG(w) : h^{2,\alpha}(\mathbb{S}^n) \rightarrow h^{0,\alpha}(\mathbb{S}^n)$ is also sectorial for any w in a neighbourhood $0 \in O_\alpha \subset h^{2,\alpha}(\mathbb{S}^n)$ and $\alpha \in (0, 1)$. Short-time existence for (6) then follows directly from Theorem 8.4.1 in [10].

Theorem 5.1. *For any $\alpha \in (0, 1)$ there are constants $\delta, r > 0$ such that if $\|u_0\|_{h^{2,\alpha}(\mathbb{S}^n)} \leq r$ then equation (6) has a unique maximal solution:*

$$u \in C([0, \delta), h^{2,\alpha}(\mathbb{S}^n)) \cap C^1([0, \delta), h^{0,\alpha}(\mathbb{S}^n)).$$

Next we see the existence of a center manifold which attracts solutions, moreover, locally this is our space of stationary solution \mathcal{S} . Let P be the spectral projection from $h^{2,\alpha}(\mathbb{S}^n)$ onto $T_0\mathcal{S}$ associated with $DG(0)$, λ_1 be the first non-zero eigenvalue of $DG(0)$, and $\psi : U \subset T_0\mathcal{S} \rightarrow (I - P)[h^{2,\alpha}(\mathbb{S}^n)]$ be the local graph function for \mathcal{S} .

Lemma 5.2. *The space \mathcal{S} is a local, invariant, exponentially attractive, center manifold for (6). In particular, there exists $r_1, r_2 > 0$ such that if $\|u_0\|_{h^{2,\alpha}(\mathbb{S}^n)} < r_2$ then there exists $z_0 \in T_0\mathcal{S}$ such that*

$$(11) \quad \|P[u(t)] - z(t)\|_{h^{0,\alpha}(\mathbb{S}^n)} + \|(I - P)[u(t)] - \psi(z(t))\|_{h^{2,\alpha}(\mathbb{S}^n)} \leq C \exp(-\omega t) \|(I - P)[u_0] - \psi(P[u_0])\|_{h^{2,\alpha}(\mathbb{S}^n)},$$

for as long as $\|P[u(t)]\|_{h^{0,\alpha}(\mathbb{S}^n)} < r_1$, where $\omega \in (0, -\lambda_1)$, C is a constant depending on ω , and

$$(12) \quad z'(t) = P \left[G \left(\eta \left(\frac{z(t)}{r_1} \right) z(t) + \psi(z(t)) \right) \right], \quad z(0) = z_0,$$

where $\eta : T_0\mathcal{S} \rightarrow \mathbb{R}$ is a smooth cut off function such that $0 \leq \eta(x) \leq 1$, $\eta(x) = 1$ if $\|x\|_{h^{0,\alpha}(\mathbb{S}^n)} \leq 1$ and $\eta(x) = 0$ if $\|x\|_{h^{0,\alpha}(\mathbb{S}^n)} \geq 2$.

Proof. The existence of a local center manifold, \mathcal{M}^c , follows from Theorem 9.2.2 in [10], where it is also shown to be a local graph over the nullspace of $DG(0)$, i.e. $T_0\mathcal{S}$. Theorem 2.3 in [12] states that \mathcal{M}^c contains all local stationary solutions, i.e. $\mathcal{S} \subset \mathcal{M}^c$, so combining these two facts we see that $\mathcal{M}^c = \mathcal{S}$. The exponential attractivity comes from Proposition 9.2.4 of [10]. \square

Using (11) evaluated at $t = 0$ we obtain

$$\begin{aligned} \|z_0\|_{h^{0,\alpha}(\mathbb{S}^n)} &\leq \|P[u_0]\|_{h^{0,\alpha}(\mathbb{S}^n)} + \|P[u_0] - z_0\|_{h^{0,\alpha}(\mathbb{S}^n)} \\ &\leq \|P[u_0]\|_{h^{0,\alpha}(\mathbb{S}^n)} + C\|(I - P)[u_0] - \psi(P[u_0])\|_{h^{2,\alpha}(\mathbb{S}^n)}, \end{aligned}$$

so since ψ is Lipschitz and P is bounded, this leads to a bound of the form $\|z_0\|_{h^{0,\alpha}(\mathbb{S}^n)} \leq C\|u_0\|_{h^{2,\alpha}(\mathbb{S}^n)}$. Therefore we can ensure that $\|z_0\|_{h^{0,\alpha}(\mathbb{S}^n)} < r_1$ by taking $\|u_0\|_{h^{2,\alpha}(\mathbb{S}^n)}$ small enough, and since $z_0 + \psi(z_0)$ defines a sphere, we see $G\left(\eta\left(\frac{z_0}{r_1}\right)z_0 + \psi(z_0)\right) = G(z_0 + \psi(z_0)) = 0$. Hence $z(t) = z_0$ is the solution to (12) and we can restate (11) as

$$(13) \quad \|P[u(t)] - z_0\|_{h^{0,\alpha}(\mathbb{S}^n)} + \|(I - P)[u(t)] - \psi(z_0)\|_{h^{2,\alpha}(\mathbb{S}^n)} \leq C \exp(-\omega t) \|(I - P)[u_0] - \psi(P[u_0])\|_{h^{2,\alpha}(\mathbb{S}^n)},$$

for as long as $P[u(t)] \in B_{r_1}(0)$. However using this bound, and our bound for z_0 , it follows that $\|P[u(t)]\|_{h^{0,\alpha}(\mathbb{S}^n)} < C\|u_0\|_{h^{2,\alpha}(\mathbb{S}^n)}$ as long as $\|P[u(t)]\|_{h^{0,\alpha}(\mathbb{S}^n)} < r_1$. By choosing $\|u_0\|_{h^{2,\alpha}(\mathbb{S}^n)}$ small enough we can therefore ensure $\|P[u(t)]\|_{h^{0,\alpha}(\mathbb{S}^n)} < \frac{r_1}{2}$ for all $t \geq 0$. Thus (13) is true for all $t \geq 0$ and this proves that $u(t)$ converges to $z_0 + \psi(z_0)$ as $t \rightarrow \infty$. This completes the proof of Theorem 1.1 since $z_0 + \psi(z_0)$ is the graph function of a sphere.

We also have the following corollary that follows by a simple continuity argument as in [5, Corollary 3.8] or [7, Corollary 4.4].

Corollary 5.3. *Let Ω_0 be a graph over a sphere with height function u_0 such that the solution, $u(t)$, to the flow (6) with initial condition u_0 exists for all time and converges to zero. Suppose further that $\left.\frac{\partial F}{\partial \kappa_i}\right|_{\kappa(X_{u(t)})} > 0$ for all $t \in [0, \infty)$ and $i = 1, \dots, n$. Then there exists a neighbourhood, O , of u_0 in $h^{2,\alpha}(\mathbb{S}^n)$, $0 < \alpha < 1$, such that for every $w_0 \in O$ the solution to (6) with initial condition w_0 exists for all time and converges to a function near zero whose graph is a sphere.*

APPENDIX A. FORM OF THE WEIGHT FUNCTION

In this appendix we determine the form the weight function must take in order to preserve the quantity \hat{V} in (5). We start by considering the linearisation of the mixed volumes. We will abuse notation and set $\hat{V}(X) = \hat{V}(X(M^n))$ and $\hat{V}_a(X) = \hat{V}_a(X(M^n))$.

Lemma A.1. *The mixed volumes have the linearisation*

$$D\hat{V}_a(X)[Y] = \int_{M^n} \hat{\Xi}_a(X) \bar{g}(Y, \nu) d\mu,$$

for $a = 0, \dots, n+1$, with $\hat{\Xi}_a$ as defined in (4).

Proof. We first note that the formula for $D\hat{V}_{n+1}(X)[Y]$ is standard.

Now we consider the linearisation of the mixed volumes with $0 \leq a \leq n$, but before starting the calculation we state some useful relations for the elementary symmetric functions:

$$(14) \quad \frac{\partial E_a}{\partial h_j^i} = g_{ik} g^{jl} \frac{\partial E_a}{\partial h_k^l},$$

$$(15) \quad \frac{\partial E_{a+1}}{\partial h_j^i} = E_a \delta_i^j - h_k^j \frac{\partial E_a}{\partial h_k^i},$$

and

$$(16) \quad h_{ij} \frac{\partial E_a}{\partial h_j^k} = h_{kj} \frac{\partial E_a}{\partial h_i^k},$$

which are all easily obtained from (9).

We can now calculate the linearisation for $a = 0, \dots, n$ using Lemma 3.1

$$\begin{aligned}
(n+1) \binom{n}{a} D\hat{V}_{n-a}(X)[Y] &= \int_{M^n} \frac{\partial E_a}{\partial h_j^i} D h_j^i(X)[Y] + \frac{E_a}{\mu} D\mu(X)[Y] d\mu \\
&= \int_{M^n} -g^{ik} \frac{\partial E_a}{\partial h_j^i} \left(\bar{g} \left(\bar{\nabla}_{T_k} \bar{\nabla}_{T_j} Y, \nu \right) + \bar{g} \left(\bar{R}(Y, T_k) \nu, T_j \right) \right) \\
&\quad - g^{ik} \frac{\partial E_a}{\partial h_j^i} h_j^l \left(\bar{g} \left(\bar{\nabla}_{T_k} Y, T_l \right) + \bar{g} \left(\bar{\nabla}_{T_l} Y, T_k \right) \right) + E_a g^{ik} \bar{g} \left(\bar{\nabla}_{T_i} Y, T_k \right) d\mu \\
&= \int_{M^n} -g^{ik} \left(h_j^l \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{\nabla}_{T_k} Y, T_l \right) + h_j^l \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{\nabla}_{T_l} Y, T_k \right) - \nabla_k \left(\frac{\partial E_a}{\partial h_j^i} \right) \bar{g} \left(\bar{\nabla}_{T_j} Y, \nu \right) \right. \\
&\quad \left. - \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{\nabla}_{T_j} Y, \bar{\nabla}_{T_k} \nu \right) + \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{R}(Y, T_k) \nu, T_j \right) - E_a \bar{g} \left(\bar{\nabla}_{T_i} Y, T_k \right) \right) d\mu \\
&= \int_{M^n} \left(h^{iq} \frac{\partial E_a}{\partial h_j^p} - g^{ip} h_j^q \frac{\partial E_a}{\partial h_j^i} - g^{iq} h_j^p \frac{\partial E_a}{\partial h_j^i} + g^{pq} E_a \right) \bar{g} \left(\bar{\nabla}_{T_p} Y, T_q \right) - g^{ik} \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{R}(Y, T_k) \nu, T_j \right) \\
&\quad - g^{ik} \nabla_j \nabla_k \left(\frac{\partial E_a}{\partial h_j^i} \right) \bar{g}(Y, \nu) - g^{ik} \nabla_k \left(\frac{\partial E_a}{\partial h_j^i} \right) \bar{g} \left(Y, \bar{\nabla}_{T_j} \nu \right) d\mu.
\end{aligned}$$

We now use Equation (14) to cancel the first two terms in the $\bar{g} \left(\bar{\nabla}_{T_p} Y, T_q \right)$ factor, and Equation (16) to alter the third term of the factor:

$$\begin{aligned}
(n+1) \binom{n}{a} D\hat{V}_{n-a}(X)[Y] &= \int_{M^n} \left(E_a g^{pq} - g^{iq} g^{pl} h_{ij} \frac{\partial E_a}{\partial h_j^l} \right) \bar{g} \left(\bar{\nabla}_{T_p} Y, T_q \right) - g^{ik} \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{R}(Y, T_k) \nu, T_j \right) \\
&\quad - g^{ik} \nabla_j \nabla_k \left(\frac{\partial E_a}{\partial h_j^i} \right) \bar{g}(Y, \nu) - g^{ik} h_j^l \nabla_k \left(\frac{\partial E_a}{\partial h_j^i} \right) \bar{g}(Y, T_l) d\mu \\
&= \int_{M^n} -g^{ip} \left(\frac{\partial E_a}{\partial h_j^i} \nabla_p h_j^k \delta_i^q - \nabla_p \left(\frac{\partial E_a}{\partial h_j^i} \right) h_j^q - \frac{\partial E_a}{\partial h_j^i} \nabla_p h_j^q \right) \bar{g}(Y, T_q) \\
&\quad - g^{ip} \left(E_a \delta_i^q - h_j^q \frac{\partial E_a}{\partial h_j^i} \right) \bar{g} \left(Y, \bar{\nabla}_{T_p} T_q \right) - g^{ik} \nabla_j \nabla_k \left(\frac{\partial E_a}{\partial h_j^i} \right) \bar{g}(Y, \nu) \\
&\quad - g^{ik} h_j^l \nabla_k \left(\frac{\partial E_a}{\partial h_j^i} \right) \bar{g}(Y, T_l) - g^{ik} \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{R}(Y, T_k) \nu, T_j \right) d\mu \\
&= \int_M \left(g^{pq} g^{ik} \frac{\partial E_a}{\partial h_j^k} \nabla_p h_{il} - g^{ip} g^{ql} \frac{\partial E_a}{\partial h_j^i} \nabla_p h_{jl} \right) \bar{g}(Y, T_q) + g^{ip} h_{pq} \frac{\partial E_{a+1}}{\partial h_q^i} \bar{g}(Y, \nu) \\
&\quad - g^{ik} \nabla_j \nabla_k \left(\frac{\partial E_a}{\partial h_j^i} \right) \bar{g}(Y, \nu) - g^{ik} \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{R}(Y, T_k) \nu, T_j \right) d\mu \\
&= \int_{M^n} -g^{ip} g^{ql} \frac{\partial E_a}{\partial h_j^i} \left(\nabla_l h_{pj} - \nabla_p h_{jl} \right) \bar{g}(Y, T_q) + h_q^i \frac{\partial E_{a+1}}{\partial h_q^i} \bar{g}(Y, \nu) \\
&\quad - g^{ik} \nabla_j \nabla_k \left(\frac{\partial E_a}{\partial h_j^i} \right) \bar{g}(Y, \nu) - g^{ik} \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{R}(Y, T_k) \nu, T_j \right) d\mu.
\end{aligned}$$

Now we use the homogeneity E_a and the Gauss-Codazzi equation:

$$\begin{aligned}
(n+1) \binom{n}{a} D\hat{V}_{n-a}(X)[Y] &= \int_{M^n} -g^{ip}g^{ql} \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{R}(T_l, T_p) T_j, \nu \right) \bar{g}(Y, T_q) + (a+1) E_{a+1} \bar{g}(Y, \nu) \\
&\quad - g^{ik} \nabla_j \nabla_k \left(\frac{\partial E_a}{\partial h_j^i} \right) \bar{g}(Y, \nu) - g^{ik} \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{R}(Y, T_k) \nu, T_j \right) d\mu \\
&= \int_{M^n} -g^{ik} \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{R} \left(Y - g^{ql} \bar{g}(Y, T_q) T_l, T_k \right) \nu, T_j \right) + (a+1) E_{a+1} \bar{g}(Y, \nu) \\
&\quad - g^{ik} \nabla_j \nabla_k \left(\frac{\partial E_a}{\partial h_j^i} \right) \bar{g}(Y, \nu) d\mu \\
&= \int_{M^n} -g^{ik} \frac{\partial E_a}{\partial h_j^i} \bar{g} \left(\bar{R} \left(\bar{g}(Y, \nu) \nu, T_k \right) \nu, T_j \right) + (a+1) E_{a+1} \bar{g}(Y, \nu) - g^{ik} \nabla_j \nabla_k \left(\frac{\partial E_a}{\partial h_j^i} \right) \bar{g}(Y, \nu) d\mu.
\end{aligned}$$

□

Corollary A.2. *If*

$$\hat{\Xi}(X) = \sum_{a=0}^{n+1} c_a \hat{\Xi}_a(X)$$

for some constants $c_a \in \mathbb{R}$, $a = 0, \dots, n+1$, where $\hat{\Xi}_a(X)$ are defined in (4), then $\hat{V}(X)$ is preserved by the flow (1).

Proof. By Lemma A.1 and linearity we have

$$(17) \quad D\hat{V}(X)[Y] = \int_{M^n} \hat{\Xi}(X) \bar{g}(Y, \nu(X)) d\mu.$$

It then follows from the form of $\hat{G}(X)$ in (2) that under (1)

$$\begin{aligned}
\frac{\partial \hat{V}}{\partial t} &= D\hat{V}(X) \left[\frac{\partial X}{\partial t} \right] \\
&= D\hat{V}(X) \left[\hat{G}(X) \nu(X) \right] \\
&= \int_{M^n} \hat{\Xi}(X) \hat{G}(X) d\mu \\
&= 0,
\end{aligned}$$

thus under this weight function we have $\hat{V}(\Omega_t) = \hat{V}(\Omega_0)$ as long as the flow exists. □

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